6 Implicatures of modified numerals

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1. Introduction

While bare numerals such as in the sentence in (1) usually receive an exact-interpretation, numerals modified by comparative *more than* or superlative *at least* such as in (2) do not give rise to such an interpretation.

(1) Three boys attended the party.

(2) a. More than three boys attended the party.
b. At least three boys attended the party.

Krifka (1999) and Fox and Hackl (2006) show that this state of affairs is surprising. If a scalar implicature were factored into the basic meaning of the sentences in (2), an exact-interpretation should result. For instance, for (2b) a scalar implicature negating the stronger alternative proposition *at least 4 boys attended the party* would have the effect that exactly three boys attended the party, where the implicature is combined with the basic interpretation of (2b).

The present chapter has three main objectives: first, I will discuss novel data that strongly suggest that the examples in (2) should be treated on a par. On the one hand, numerals modified by *more than* are shown to not have scalar implicatures in exactly those environments where numerals modified by *at least* lack them as well. On the other hand, it is furthermore shown that if one of the two types of modified numerals occurs embedded in an environment where a scalar implicature is nevertheless triggered, the same holds for the other. Second, by investigating these details, I suggest a new descriptive generalization which any account of (2) should incorporate. Third, I argue that currently existing accounts for the facts in (2) are not completely satisfying empirically. The fully predictive theory by Fox and Hackl (2006), in particular, unfortunately only covers cases with comparative *more than* going against the picture argued for in this paper. Because of this defect, I turn to the discussion of two closely related approaches. Both stipulate certain alternatives that are put to use by the implicature generating mechanism, and in both accounts the alternatives are of such a form that negating one of them contradicts the negation of another one. Thus neither is negated, and no scalar implicature is derived. On both accounts it moreover follows that when embedded under certain operators the modified numeral can introduce an implicature. But I also show that the accounts of the facts in (2) both have certain problems. First, I discuss accounts making use of *exactly n* alternatives for modified numerals (for instance, Spector 2005). Both the neo-Gricean version and the grammatical view of scalar implicatures are shown to run into problems: they predict truth conditions that are too strong. Another theory making use of non-monotonic alternatives is suggested. It will, however, be seen that unless a way is found to contextually restrict the set of available alternatives, truth conditions are predicted that are again not in accordance with our intuitions.

The paper is structured as follows: in Section 2 I review the neo-Gricean theory of scalar implicatures and the problem presented by the data in (2). Section 3 discusses some novel data and presents evidence that the facts in (2) should be handled by one theory by investigating in detail under which circumstances scalar implicatures appear for modified numerals. Section 4 proposes a novel generalization. Section 5 argues that existing accounts run into problems with the data discussed in the preceding sections. In Section 6, I present the two related approaches relying on the structure of the alternatives to account for the problem. Section 7 concludes the paper.

2. The problem

2.1 Gricean reasoning and bare numerals

It has long been noted that numerals should not receive an exact-interpretation but rather an at-least-interpretation as their basic non-strengthened denotation.[1] The exact-interpretation intuitively associated with sentences embedding a numeral should be treated as resulting from an inference, in particular a scalar implicature based on the hearer's reasoning employing Grice's 1975 maxim of quantity (cf. Gazdar 1979; Horn 1972; Levinson 1983, a.m.o.). The maxim of quantity essentially states that if two sentences φ and ψ are both relevant in the conversation, and φ is more informative than ψ – i.e., the denotation of φ asymmetrically entails the one of ψ – and the speaker believes both φ and ψ to be true, the speaker should choose φ over ψ.

Consider a current version of this theory, often called neo-Gricean. (3) has an exact-inference. In particular it has the inference associated with it that Jack did not read (at least) four books (non-entailment inferences are represented by $\rightsquigarrow$ throughout the chapter) or any larger number of books. This taken together with the basic inference obtained from the assertion that Jack read at least three books derives the strengthened inference that Jack read exactly three books.

(3) Jack read three books.
⇝ Jack did not read four books or more.

According to Horn (1972) scalar items are lexically collected in sets ordered by informativeness. The following are relevant Horn-sets {or, and}, {some, all}, {1, 2, 3, 4, …}. From these sets of scalar alternatives one derives a set of alternative sentences *Alt*(S) for a given sentence S by replacing the members of a given Horn-set with each other (Sauerland 2004). The neo-Gricean version of the maxim of quantity can then be stated as follows:

(4) Maxim of quantity
In world w of evaluation if φ and ψ are both relevant in the conversation, $\varphi \subset \psi$, $\varphi \in \mathrm{Alt}(\psi)$, and the speaker believes both $\varphi(w) = 1$ and $\psi(w) = 1$, the speaker should choose φ over ψ.

For (3) this means that the hearer reasons as follows: the speaker believes that Jack read at least three books, which corresponds to the literal interpretation of (3). This is the basic inference in (5a), and it follows from Grice's maxim of quality, which says that a speaker should not present something as true that she believes to be false. Moreover, *Alt* (3) includes the stronger propositions that Jack read four books, that Jack read five books and so on. Since these would also have been relevant to the topic of conversation, the speaker should have chosen one of these stronger propositions if she believed them to be true. But the speaker did not choose any of them. The hearer therefore concludes that the speaker does not believe any of these to be true (in particular, (5b) follows). Sauerland (2004: 383) argues that if nothing in the context precludes it, the hearer will further strengthen (5b) to (5c), that is, the speaker does not believe that Jack read four books. In particular, this process of strengthening is allowed to apply when no contradiction results.[2] In the present example strengthening (5b) to (5c) does not lead to a contradiction. The stronger inference (5c), together with the speaker's belief that the basic meaning of (3) is true, leads the hearer to conclude (5d), which is equivalent to the strengthened inference that the speaker believes that John read exactly three books (in the following $B_S p$ denotes *'the speaker believes p')*.

(5) a. *Basic inference of* (3): B_S(that Jack read at least 3 books)
b. *Scalar inference of* (3): $\neg B_S$(that Jack read at least 4 books)
c. *Strengthened scalar inference of* (3): $B_S \neg$(that Jack read at least 4 books)
d. *Strengthened inference of* (3):
B_S(that Jack read at least 3 books) $\wedge$ $B_S \neg$(that Jack read at least 4 books)

It can now be seen why Horn-alternatives are necessary. Without such stipulated alternatives one would derive for (3) also an alternative proposition stating that Jack read three books but not four. This alternative is also strictly stronger than the basic interpretation of (3), and therefore by the same reasoning process as discussed above, the inference that the speaker does not believe that Jack read three books but not four would be drawn, i.e., the speaker does not believe that Jack read exactly three books. But if that inference were derived, the strengthened scalar inference in (5c) could not be drawn. Together with the basic inference it would contradict the inference that the speaker does not believe that Jack read exactly three books. In fact, this latter inference could also not be strengthened to the inference that the speaker believes that Jack did not read exactly three books. Together with the basic inference it would entail that Jack read more than three books. But this contradicts the basic scalar inference in (5b). In other words, no strengthened inference could be drawn at all. For this reason the problem that would arise without Horn-alternatives is termed the symmetry problem. No strengthened scalar inference as in (5c) could ever be drawn.[3]

Evidence for the claim that numerals have the at-least-interpretation as their denotation and moreover form a Horn-set can be attained when embedding them in downward-entailing (DE) contexts. It is known that implicatures tend to disappear in such environments (cf. Chierchia 2004; Gazdar 1979). This makes the prediction that the exact-interpretation of *three* in (3) should disappear or be weakened when the sentence is embedded in a DE context if the exact-interpretation is derived via an implicature. If *three*, however, had the exact-interpretation as its denotation, DE contexts should not affect the availability of the exact-interpretation. Embedding (3) in DE contexts such as clausal negation (6a), the antecedent of a conditional (6b), or the restrictor of a universal quantifier (6c) suggests that the former option is correct because in these contexts the exact-interpretation of the numeral *three* is systematically suspended.[4]

(6) a. Jack didn't read three books.
$\rightsquigarrow$ Jack didn't read four books.
b. If Jack read three books, he can participate in the course.
$\rightsquigarrow$ If Jack read four books, he can participate
c. Everyone who read three books can participate.
$\rightsquigarrow$ Everyone who read four books can participate.

2.2 The absence of implicatures with modified numerals

Krifka (1999: 258) notes that sentences embedding *at least n* – that is, numerals *n* modified by the superlative *at least* – do not lead to an implicature that *not at least n + 1*. If (7) licensed such an inference, it would be taken to imply that Jack read exactly three books.

(7) Jack read at least three books.
$\not\rightsquigarrow$ Jack did not read at least four books.

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Fox and Hackl (2006: 540) note that numerals *n* modified by the comparative *more than* similarly do not give rise to an implicature that *not more than n + 1*. Otherwise (8) would imply that Jack read exactly four books.

(8) Jack read more than three books.
$\not\rightsquigarrow$ Jack did not read more than four books.

This behavior is unexpected. If scalar implicatures are derived as argued, then (7) and (8) should have the unobserved interpretations, and if bare numerals form a Horn-set even more so. Two types of approaches have been suggested to deal with these data to my knowledge. On the one hand, *at least* has been claimed by Krifka (1999) and Nouwen (2008) to behave similar to *only* in that it consumes the scalar alternatives in its scope (Rooth 1992) and leaves no alternatives to be evaluated for higher operators, i.e., the implicature generating mechanism has no alternatives to operate on because *at least* uses up the alternatives of the numeral. On the other hand, Fox and Hackl (2006) propose a detailed theory of why implicatures are missing for *more than n*, making use of the assumption that all measurement scales in natural language are dense. Their argument does not carry over to *at least n*. Therefore another theory, for instance one like the first one mentioned, is needed on top of Fox and Hackl's (2006) one.[5] In the following section I will discuss data that suggest that this position is untenable: implicatures appear under certain conditions for both *more than n* and *at least n*. And crucially these conditions are the same for both types of modified numerals.[6] This does not follow from Fox and Hackl's (2006) account, and is also unexpected if *at least* consumes the alternatives of the numeral. I will return to their account in Section 5, where in addition I point out a false empirical prediction.

3. Characterization of the problem

3.1 Fox and Hackl's 2006 observation and a novel observation

Modified numerals do sometimes have scalar implicatures: Fox and Hackl (2006: 544) make the crucial observation that universal modals reintroduce the implicature of *more than n* (9), whereas existential modals do not do so (10). In other words, in case *more than n* is embedded under a universal modal, an exact interpretation becomes available, but not if it is embedded under an existential modal.

(9) Jack is required to read more than three books.
$\rightsquigarrow$ Jack is not required to read more than four books.

(10) Jack is allowed to read more than three books.
$\not\rightsquigarrow$ Jack is not allowed to read more than four books.

From this observation Fox and Hackl proceed to develop an interesting analysis, which will be discussed in detail in Section 5 below. As will be seen there, their analysis correctly captures the fact that *more than n* only triggers a scalar implicature when embedded under a universal modal. They do not intend their analysis to be applied to numerals modified by the superlative *at least*. That is, the non-availability of a scalar implicature in (7) should have different roots than the one in (8).

Fox and Hackl do not note that *at least n* shows a behavior parallel to the one of *more than n* when it comes to the appearance of scalar implicatures under embedding. Only in a sentence where *at least n* is embedded under a universal modal does the exact interpretation become possible:

(11) Jack is required to read at least three books.
$\rightsquigarrow$ Jack is not required to read at least four books.

(12) Jack is allowed to read at least three books.
$\not\rightsquigarrow$ Jack is not allowed to read at least four books.

The parallelism observed between *more than n* and *at least n* is interesting insofar as it might suggest that a unified approach to the lack of scalar implicatures in (7) and (8) could after all be called for, contrary to Fox and Hackl's (2006) suggestions. In the following, I investigate when scalar implicatures appear with modified numerals and when they do not do so strengthening this initial suspicion.

3.2 When are scalar implicatures generated?

Universal modals allow for the appearance of scalar implicatures with both *more than n* and *at least n*. This leads us to expect that conditionals should also license scalar implicatures with modified numerals under Stalnaker's (1975) analysis and the approach advocated by Kratzer (1979, 1986) where conditionals are analyzed with implicit universal quantification. This is indeed the case. (14a) implies that there is a world where John is well prepared and he read exactly four books, whereas (14b) implies that there is a world where John is well prepared and he read exactly three books.[7]

(13) a. If John is well prepared, he read more than three books.
$\rightsquigarrow$ Not in all worlds where John is well prepared he read more than four books.
b. If John is well prepared, he read at least three books.
$\rightsquigarrow$ Not in all worlds where John is well prepared he read at least four books.

Embedding the modified numeral in the restrictor of the universal quantifier does not affect the availability of the implicature, which has not been pointed out to my knowledge. Only the entailment pattern is reversed, as the restrictor of a conditional is DE and DE contexts change entailment patterns. Therefore the stronger alternatives to, say, *three* are now all the numerals smaller than *three*. (14a) suggests that there is a world where John read exactly three books and he is not well prepared. Similarly, in (14b) there is a world where John read exactly two books and he is not well prepared.

(14) a. If John read more than three books, he is well prepared.
$\rightsquigarrow$ Not in all worlds where John read more than two books, he is well prepared.
b. If John read at least three books, he is well prepared.
$\rightsquigarrow$ Not in all worlds where John read at least two books he is well prepared.

Fox and Hackl (2006: 576) note that not only do universal modals allow for the appearance of scalar implicatures with *more than n*, universal quantifiers over individuals do so as well (15a). Thai is, (15a) does have the inference that at least one person wrote exactly four books. I add that the same holds for *at least n*, (15b). It suggests that someone wrote exactly three books.[8]

(15) a. Everyone wrote more than three books.
$\rightsquigarrow$ Not everyone wrote more than four books.
b. Everyone wrote at least three books.
$\rightsquigarrow$ Not everyone wrote at least four books.

As with conditionals, we find that parallel behavior can be observed by embedding the modified numerals in the restrictor of the universal quantifier. (16a) and (16b) suggest that there is someone who read exactly three books and someone who read exactly two books, respectively, who did not participate.

(16) a. Everyone who read more than three books participated.
$\rightsquigarrow$ Not everyone who read more than two books participated.
b. Everyone who read at least three books participated.
$\rightsquigarrow$ Not everyone who read at least two books participated.

Furthermore, a DE quantifier like *no one* has the same effect, with the caveat that *at least* is not perfect in the scope of negation, presumably due to the fact that *at least n* is a positive polarity item and can therefore not occur under negation.[9] It seems, however, that *at least* is more acceptable in the restrictor of the negative quantifier, which would suggest that it is a local positive polarity item. (17a) implies that someone wrote exactly three books, and (17b) that someone wrote exactly two books. (18a) implies that someone who read exactly three books participated, and (18b) suggests that someone who read exactly two books participated.

(17) a. No one wrote more than three books.
$\rightsquigarrow$ Someone wrote more than two books.
b. ?No one wrote at least three books.
$\rightsquigarrow$ Someone wrote at least two books.

(18) a. No one who read more than three books participated.
$\rightsquigarrow$ Someone who read more than two books participated.
b. No one who read at least three books participated.
$\rightsquigarrow$ Someone who read at least two books participated.

Comparative quantifiers like *more than half of the NPs* also allow for the scalar implicature associated with modified numerals to be generated. (19a) implies that some students wrote exactly four books, while (19b) implies that some of the students wrote exactly three books.

(19) a. More than half of the students wrote more than three books.
$\rightsquigarrow$ At most half of the students wrote more than four books
b. More than half of the students wrote at least three books.
$\rightsquigarrow$ At most half of the students wrote at least four books

Moreover, we observe that distributive conjunctions of individuals show a parallel behavior. The examples in (20) suggest that one of John and Mary wrote exactly four books or exactly three books.

(20) a. John and Mary both wrote more than three books.
$\rightsquigarrow$ Not both of John and Mary wrote more than four books.
b. John and Mary both wrote at least three books.
$\rightsquigarrow$ Not both of John and Mary wrote at least four books.

The facts about *no one*, the comparative *more than half*, and distributive conjunction have not been noticed to my knowledge. Let me now turn to discussion of environments where scalar implicatures are not generated.

3.3 When are scalar implicatures not generated?

We already know that an exact-interpretation is not generated when modified numerals occur non-embedded or under an existential modal. To this list we can add embedding under sentential negation. For discussion of the sentences to follow it is important to have in mind the surface scope interpretation of the relevant sentences. Since the negation is DE, the strength of the alternatives is, of course, reversed. Consider (21a) under this aspect first. It does not seem to imply that Jack read exactly three books, which would be expected if the implicature noted were available. Notice moreover that *not more than n* is equivalent to *at most n*, where *at most* is DE. In other words, (21a) is equivalent to (21b), and it is clear that (21b) does not have the exact-interpretation either.

(21) a. Jack didn't read more than three books.
$\not\rightsquigarrow$ Jack read more than two books.
b. Jack read at most three books.
$\not\rightsquigarrow$ Jack didn't read at most two books.

Similarly, (22a) does not have the exact-interpretation, i.e., it does not have the interpretation that Jack read exactly two books. But (22a) is slightly degraded due to the positive polarity status of *at least n*.[10] The situation is, however, clearer with (22b). Note that *not at least n* is equivalent to *fewer than n*, where *fewer than* is of course DE. (22b) clearly does not have the exact-interpretation.

(22) a. ?Jack didn't read at least three books.
$\not\leadsto$Jack read at least two books.
b. Jack read fewer than three books.
$\not\leadsto$Jack didn't read fewer than two books.

In other words, the situation observed is parallel to the one discussed for positive environments in that for both types of modified numerals no scalar implicature is generated. And importantly, sentential negation does not pattern with negative quantifiers like *no one* discussed in (17) and (18), which *do* allow scalar implicatures to be generated for modified numerals embedded under them. This difference has not been discussed in the literature so far.

Because of the fact that modified numerals embedded under existential modals do not generate scalar implicatures, we expect that no implicature is generated either when it is embedded in the antecedent of a conditional with an existential modal. This is the case, as (23) shows:

(23) a. If Jack read more than three books, he is allowed to participate.
$\not\leadsto$There is no world where Jack read more than four books and he participates.
b. If Jack read at least three books, he is allowed to participate.
$\not\leadsto$There is no world where Jack read at least four books and he participates.

Fox and Hackl (2006: 576) also note that existential quantifiers ranging over individuals do not license a scalar implicature for *more than n* (24a). Let me add that we can make the same observation for *at least n* (24b). From what we have seen so far we also expect that modified numerals embedded in the restrictor of an existential quantifier should behave in exactly the same way. This is borne out as shown in (25).

(24) a. Someone wrote more than three books.
$\not\leadsto$No one wrote more than four books.
b. Someone wrote at least three books.
$\not\leadsto$No one wrote at least four books.

(25) a. Someone who read more than three books participated.
$\not\leadsto$No one who read more than four books participated .
b. Someone who read at least three books participated.
$\not\leadsto$No one who read (at least) four books participated.

Finally, given that conjunction and disjunction can be seen as a form of universal and existential quantification, respectively, and that embedding of

Table 6.1. *Abbreviations: +: scalar inference possible; −: scalar inference impossible; root: non-embedded context; : sentential negation; ∀: universal quantifier; □: universal modal; ∃: existential quantifier; ◇: existential modal; ∃: negative quantifier; >1/2: comparative quantifier; ∧: conjunction; ∨: disjunction. The number below each abbreviation in the top row indicates the section where the relevant data are discussed.*

Context / Modified Numeral	root (3.2)	¬ (3.3)	∀ (3.2)	□ (3.2)	∃ (3.3)	◇ (3.3)	¬∃ (3.2)	>1/2 (3.2)	∧ (3.2)	∨ (3.3)
more than *n*	−	−	+	+	−	−	+	+	+	−
at least *n*	−	−	+	+	−	−	+	+		−

modified numerals under conjunction does allow for scalar inferences, we expect that such inferences are not generated by embedding under a disjunction of individuals. Again, this expectation is confirmed by the novel data in (26).

(26) a. John or Mary wrote more than three books.
$\not\rightsquigarrow$Neither John nor Mary wrote more than four books.
b. John or Mary wrote at least thee books.
$\not\rightsquigarrow$Neither John nor Mary wrote at least four books.

3.4 The distribution of scalar implicatures for modified numerals

Given the discussion in this section an empirical picture summarized by Table 6.1 presents itself. Here all the embeddings for modified numerals discussed above are collected. I do not discriminate between restrictor and scope of quantifiers, as we did not find any difference in the behavior of these two environments. Moreover, conditionals are not listed as separate environments. Their behavior is completely dependent on the type of modal chosen.

Fox and Hackl (2006) already observed that universal quantifiers and existential quantifiers allow for the generation of scalar implicatures associated with *more than n*. The novel observations, which will be seen to be crucial later on, are the following: first, sentential negation and negative quantifiers behave differently. Only the latter allow for scalar implicatures of embedded modified numerals, and therefore we cannot conclude that negation in general blocks such inferences. Second, conjunction and disjunction of individuals also differs along the universal–existential divide in that only the former constitute environments for scalar implicatures of modified numerals. But crucially third, comparative quantifiers like *more than half* behave differently from ordinary existential quantifiers in that they do allow for scalar implicatures. In other words, it is not the case that existential quantification per se blocks scalar inferences of modified numerals. And finally, superlative *at least n* is not inherently different from comparative *more than n* when it comes to scalar implicatures. The two types of modified numerals behave in a completely parallel fashion, at least with regards to the environments summarized above. This is, of course, not to say that they might not differ in other respects.

4. Towards a generalization

On the basis of the empirical discussion in the preceding section, I now point out an almost obvious difference between examples with modified numerals showing a scalar implicature and examples not doing so. I submit that this fact cannot be a coincidence and propose that it should be the backbone of the descriptive generalization that we wish to account for.

There is a feature that all the examples in Subsection 3.2 share, and that is absent from all the examples surveyed in Subsection 3.3: notice that in the cases where no scalar implicature is generated in a sentence $Op\varphi$ – with Op being a possibly null operator – the interpretation that would have resulted if an implicature had been generated would be entailed by the *basic* interpretation of $Op\psi$, which is just like $Op\varphi$ except that the modified numeral has been replaced by an *exactly n* expression. This is obvious for the cases not involving any quantification. But for the examples involving existential quantifiers this is also the case. Consider (27a) and (27b) again, repeated from (24) above. Factoring in the implicature would have the consequence that (27a) would have as its interpretation the proposition that someone wrote more than three books and no one wrote more than four books. (27b) would have the interpretation that someone wrote three books and no one wrote four books. In other words, in each case the strengthened interpretation of the sentence is entailed by the basic interpretation of a minimally differing sentence with an *exactly n* expression.

(27) a. Someone wrote more than three books.
$\not\rightsquigarrow$No one wrote more than four books.
b. Someone wrote at least three books.
$\not\rightsquigarrow$No one wrote (at least) four books.

The cases in (27) differ from the ones in (28), repeated from (15) above. The strengthened interpretation in (28a) is not entailed by the proposition that everyone wrote exactly four books. And the strengthened interpretation of (28b) is not entailed by the proposition that everyone wrote exactly three books. Both of these propositions would be the basic interpretations of sentences differing from the ones in (28) only in the fact that the modified numeral has been replaced by an *exactly n* expression.

(28) a. Everyone wrote more than three books.
$\rightsquigarrow$ Not everyone wrote more than four books.
b. Everyone wrote at least three books.
$\rightsquigarrow$ Not everyone wrote four books.

Parallel facts can be shown to hold for the other operators embedding modified numerals. It is thus natural to conclude that a scalar implicature for a modified numeral like *more than n* or *at least n* is generated whenever this does not partially undo the truth-conditional contribution of *more than* or *at least* and reduces it to essentially the contribution that *exactly* would make for all worlds or individuals quantified over by the embedding operator if such an operator exists. In other words, the scalar implicature is generated as long as a world or individual can remain after strengthening the basic meaning of the sentence such that *more than* or *at least* makes a truth-conditional difference for that world or individual when compared to *exactly*. Simply put, the potential strengthened interpretation of a sentence with a modified numeral in it and all its implicatures factored in ($[[]]^S$) should not be entailed by the basic interpretation of an alternative sentence with an *exactly n* expression in it:

(29) **Generalization**
For any φ with a modified numeral α *n* in it, a scalar implicature for α *n* is generated iff $[[\psi]]$ does not entail $[[\varphi]]^S$, where ψ is just like φ except that α *n* is replaced by *exactly n+1, exactly n−1*, or *exactly n*.

Given this generalization it is furthermore natural to expect that the exact-interpretation of numerals should play a role in the account of the absence of scalar implicatures for modified numerals. Two such possible approaches will be discussed in Section 6. But before doing so, I will discuss in some more depth previous analyses and point out some problems for them.

5. "Density" plus something else

In this section I discuss an approach to numerals modified by comparative *more than*, which crucially relies on the property of density. It will be seen that another ingredient is needed in order to also account for numerals modified by superlative *at least*. Two things, will be argued: first, suggestions found in the literature as to what this additional ingredient might be are not on the right track. Second and more importantly, an approach relying on density makes wrong predictions when it comes to embedding under negation.

Fox and Hackl (2006) argue for the following principle (also cf. Fox 2007b; Nouwen 2008):

(30) **The Universal Density of Measurements (UDM)**
Measurement scales needed for natural language semantics are *always* dense. (Fox and Hackl 2006: 542)

That is, scales in natural language are set up in such a way that for any two degrees there are infinitely many degrees between the former two. In other words, for any n and $n + \varepsilon$ there is a degree $n + \delta$ such that $n < n + \delta < n + \varepsilon$. This assumption has the consequence that the Horn-set for numerals is not the scale of cardinal numbers $\{1, 2, 3, \ldots\}$ but rather a dense scale. What does this mean for the data at hand? Assume with Hackl (2000) that comparative quantifiers contain a silent *many*. (31) on its basic interpretation then states that the comparative quantifier *more than two girls* applies to a set of degrees d such that there is an individual x that John kissed and the cardinality of x is larger than d. This is equivalent to (32a). It is therefore required that John kissed more than two girls, i.e., he kissed $2 + \varepsilon$ girls. But since the number scale is dense, it follows that he also kissed $2 + \varepsilon/2$ girls. The implicature generating mechanism now negates all stronger alternatives to (32a).[11] The potential implicature states that for any degree d larger than 2 it is not the case John kissed more than d-many girls. So he did not kiss more than $n + \varepsilon/2$ girls. But this is contradictory because by the basic interpretation of the assertion in (32a) it is required that John kissed $2 + \varepsilon$ girls. Contradictory implicatures are not generated. Thus (31) is predicted to not have an exact-interpretation.

(31) John kissed more than two girls.

(32) a. $[[(31)]] = \exists x[|x| > 2$ and John kissed x and x is a girl$]$
b. $\forall d[d > 2 \rightarrow \neg($John kissed more than d-many girls$)]$

This type of explanation does not carry over to examples with modification by *at least*. The basic interpretation of (33) states that there is an individual of at least three girls such that John kissed that individual. This is equivalent to saying that John kissed 3 girls or $3 + \varepsilon$ girls. The potential implicature for (33) states that for all degrees d larger than 3 it is not the case that John kissed at least d-many girls. In other words, it cannot be the case that John kissed (at least) $3 + \varepsilon$ girls. But this implicature is consistent with the assertion in (34a), and we get the strengthened interpretation $[[\ ]]^S$ in (34c), where the negated stronger alternatives have been factored into the basic meaning, saying that John kissed exactly three girls.

(33) John kissed at least three girls.

(34) a. $[[(33)]] = \exists x[|x| \geq 3$ and John kissed x and x is a girl$]$
b. $\forall d[d > 3 \rightarrow \neg($John kissed at least d-many girls$)]$
c. $[[(33)]]^S = \exists x[|x| = 3$ and John kissed x and x is a girl$]$

Thus clearly density only predicts the absence of any scalar implicature for sentences with *more than n* in them. Fox and Hackl (2006: fn. 4) suggest that *at least n* is different from *more than n* and the unavailability of the implicature with the former should be accounted for independently.[12] Nouwen (2008) for related reasons suggests to follow Krifka (1999) in the assumption that *at least* consumes the alternatives of the numeral, similar to the effects of the ~ operator argued for by Rooth (1992) (also cf. Beck 2006), so that the implicature generating mechanism has no alternatives to work on anymore. Is such an assumption realistic?

5.1 *Similarities between* at least n *and* more than n

As discussed in Subsection 3.1, Fox and Hackl (2006: 544) observe that universal modals reintroduce the implicature of *more than n*, as in (35) repeated from (9) above. Existential modals, however, do not do so, as in (36) repeated from (10) above.

(35) Jack is required to read more than three books.
$\rightsquigarrow$ Jack is not required to read more than four books.

(36) Jack is allowed to read more than three books.
$\not\rightsquigarrow$ Jack is not allowed to read more than three books.

The difference between embedding *more than n* under a universal modal and under an existential one follows naturally under Fox and Hackl's (2006) proposal. Consider first (35). The basic meaning in (37a) states that in all worlds Jack reads more than three books. This means that in all worlds *w* there is a degree ε such that Jack reads $3 + \varepsilon$ books in w and that therefore Jack reads $3 + \varepsilon/2$ books in *w*. The potential implicature in (37b) states that for each degree *d* greater than *3* there is a world *w* such that Jack does not read more than *d*-many books in *w*. If we assume that the modal base over which the modals quantify are the set of worlds corresponding to the dense degrees *d* greater than *3*, then it follows that for each *d* there is a world where Jack reads at most *d*-many books. But the basic meaning in (37a) and the implicature in (37b) are consistent. It is possible that Jack reads more than three books in each world and moreover that for each degree *d* larger than *3* there is a world where Jack reads at most *d*-many books. The strengthened interpretation in (37c) is derived: there is a world where Jack does not read more than four books.[13]

(37) a. $[[(35)]] = \forall w \exists x[|x| > 3$ and Jack reads x and x is a book in $w]$
b. $\forall d[d > 3 \rightarrow \neg\forall w$[Jack reads more than d-many books in w]
$= \forall d[d > 3 \rightarrow \exists w[\neg$(Jack reads more than d-many books in w)]
c. $[[(35)]]^S = \forall w \exists x[|x| > 3$ and Jack reads x and x is a book in $w]$ and $\exists w \neg \exists x[|x| >$ 4 and Jack reads x and x is a book in $w]$

The exact interpretation for (36), however, is not allowed. The basic interpretation in (38a) states that in some world Jack read more than three books, i.e., there is a world *w* where Jack reads $3 + \varepsilon$ books, and thus Jack reads $3 + \varepsilon/2$ books in *w*. The potential implicature requires that for all degrees *d* greater than *3* there is no world where Jack read more than *d*-many books, (38b). In particular, it requires that there is no world where Jack read more than $3 + \varepsilon/2$ books. But this contradicts the basic meaning stating that in *w* Jack read $3 + \varepsilon$ books, and therefore strengthening does not apply.

(38) a. $[[(36)]] = \exists w \exists x[|x| > 3$ and Jack read x in w and x is a book in $w]$
b. $\forall d[d > 3 \rightarrow \neg\exists w$[Jack reads more than d-many books in w]]

But recall that we saw in Section 3 that *at least n* shows a completely behavior parallel to the one of *more than n* when it comes to the appearance of scalar implicatures in embedded contexts. In particular, if *at least n* is embedded under a universal modal, the exact interpretation becomes possible, as in (39) repeated from (11). But this is not the case if *at least n* is embedded under an existential modal, as in (40) repeated from (12).

(39) Jack is required to read at least three books.
$\rightsquigarrow$ Jack is not required to read (at least) four books.

(40) Jack is allowed to read at least three books.
$\not\rightsquigarrow$Jack is not allowed to read (at least) four books.

I said that Fox and Hackl (2006) do not intend their proposal to capture numerals modified by *at least n*. Nevertheless it is interesting to note for further discussion what their account would predict for the data in (39) and (40). Consider first (39). The basic meaning says that in all worlds *w* there is a degree ε such that Jack reads *3* books or $3 + \varepsilon$ books in *w*, (41a). The potential implicature states that for each degree *d* greater than *3* there is a world *w* such that Jack reads fewer than *d*-many books in *w*. If the modal base is again the set of worlds corresponding to the dense degrees greater than or equal to *3*, the basic meaning and the implicature are consistent: it is possible that in each world Jack reads three books or more while there still being for each degree *d* greater than *3* a world where he reads fewer than *d*-many books, as long as he reads at least three books in each of these worlds. But then it follows that there is no degree *d* greater than *3* such that Jack must read at least *d*-many books. Strengthening can apply, (41c).

(41) a. $[[(39)]] = \forall w \exists x[|x| \geq 3$ and Jack reads x in w and x is a book in $w]$
b. $\forall d[d > 3 \rightarrow \neg\forall w$[Jack reads at least d-many books in w]]
$= \forall d[d > 3 \rightarrow \exists w[\neg$(Jack reads at least d-many books in w)]]
c. $[[(39)]]^S = \forall w \exists x[|x| \geq 3$ and Jack reads x and x is a book in $w]$ and $\exists w \neg \exists x[|x| > 3$ and Jack reads x and x is a book in $w]$

What about (40)? Its basic meaning states that there is a world w such that Jack reads *3* books or $3 + \varepsilon$ books in w. But the implicature would state that for all degrees d greater than *3* there is no world such that Jack reads at least d-many books in w. It follows that Jack cannot read $3 + \varepsilon$ books in w. Nevertheless the implicature is consistent with the basic interpretation, as the strengthened meaning in (42c) would imply that Jack is only allowed to read exactly three books, contrary to fact.

(42) a. $[[(40)]] = \exists w \exists x[|x| \geq 3$ and Jack reads x in w and x is a book in $w]$
b. $\forall d[d > 3 \rightarrow \neg\exists w$[Jack reads at least d-many books in w]]
c. $[[(40)]]^S = \exists w \exists x[|x| = 3$ and Jack reads x and x is a book in $w]$

The fact that the density-based approach predicts an implicature for (40) is not in itself problematic. We already know that the account is not meant to be applied to data with numerals modified by *at least*. But it must be noted that the parallel behavior of *at least n* and *more than n* makes us suspect that the density-based account misses a generalization. In other words, we have cast some initial doubts on the density-based approach to missing implicatures for *more than n*. Moreover, the observation that *at least n* does sometimes have a scalar implicature as in (39) is at odds with Krifka's (1999) and Nouwen's (2008) assumptions that *at least* consumes the alternatives of the numeral. If the latter were the case, implicatures should also be unavailable for (39). But then it follows that this analysis cannot be the independent theory needed by Fox and Hackl (2006) in order to make the correct predictions with respect to *at least n*.[14]

I will now show that the density approach runs into more severe problems once embedding under negation is considered.

5.2 *Embedding under negation*

Remember from Subsection 3.3 that both types of modified numerals do not generate scalar implicatures when embedded under sentential negation. Furthermore recall that due to the equivalence between *not more than* with *at most n*, on the one hand, and between *not at least n* and *fewer than n*, on the other hand, we also do not observe scalar inferences in these latter cases. (43) and (44) are repeated from (21) and (22), respectively.

(43) a. Jack didn't read more than three books.
$\not\rightsquigarrow$Jack read more than two books.
b. Jack read at most three books.
$\not\rightsquigarrow$Jack didn't read at most two books.

(44) a. ? Jack didn't read at least three books.
$\not\rightsquigarrow$Jack read at least two books.
b. Jack read fewer than three books.
$\not\rightsquigarrow$Jack didn't read fewer than two books

What does the density-based approach predict for the data above? Let us start by considering (43a). The basic meaning states that Jack read 3 or fewer books, (45a), i.e., Jack read *3* books or Jack read $3 - \varepsilon$ books. The implicature says that for all degrees d smaller than *3* it is not the case that Jack didn't read more than d-many books, i.e., Jack read more than d-many books. Therefore Jack read more than $3 - \varepsilon$ books. But the implicature is consistent: it follows that Jack read exactly three books, (45c), contrary to fact.

(45) a. $[[(43a)]] = \neg\exists x[|x| > 3$ and Jack read x and x is a book$]$
b. $\forall d[\mathrm{d} < 3 \rightarrow$ Jack read more than d-many books$]$
c. $[[(43a)]] = \exists x[|x| = 3$ and Jack read x and x is a book$]$

The reasoning for (43b) is, of course, almost parallel. The basic meaning states that Jack read *3* books or Jack read $3 - \varepsilon$ books, (46a), while the implicature says that for all degrees d smaller than *3* it is the case that Jack read more than d-many books, (46b). Therefore Jack read more than $3 - \varepsilon$ books. So Jack read exactly three books, (46c).

(46) a. $[[(43b)]] = \exists x[|x| \leq 3$ and Jack read x and x is a book$]$
b. $\forall d[d < 3 \rightarrow \neg($Jack read at most d-many books$)]$
c. $[[(43b)]]^S = \exists x[|x| = 3$ and Jack read x and x is a book$]$

Consider now (44a). Its basic interpretation is that Jack read fewer than 3 books, (47a). That is, Jack read $3 - \varepsilon$ books and therefore it is not the case that he read $3 - \varepsilon/2$ books. But the potential implicature states that for all degrees d smaller than 3 Jack read at least d-many books (47b), from which it follows that Jack read at least $3 - \varepsilon/2$ books. This is a contradiction and the implicature is not generated.

(47) a. $[[(44a)]] = \neg\exists x[|x| \geq 3$ and Jack read x and x is a book$]$
b. $\forall d[d < 3 \rightarrow$ Jack read at least d-many books$]$

Again, the reasoning for (44b) is parallel to the one just given. The basic meaning says that Jack read $3 - \varepsilon$ books and therefore he did not read $3 - \varepsilon/2$ books, (48a). The implicature would require that he read at least $3 - \varepsilon/2$ books (48b), which would be contradictory.[15]

(48) a. $[[(44b)]] = \exists x[|x| < 3$ and Jack read x and x is a book$]$
b. $\forall d[d < 3 \rightarrow \neg($Jack read fewer than d-many books$)]$

The situation observed is problematic for Fox and Hackl's (2006) account. Recall that according to this approach, the exact-interpretation for numerals modified by comparative *more than* is absent due to the hypothesized density of measurement scales. The absence of such an interpretation for numerals modified by superlative *at least* is in need of an independent explanation, as density alone would predict precisely that interpretation. As shown in the present subsection, however, the

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pattern switches when the modified numerals are embedded under negation: on the one hand, the absence of an exact-interpretation for *more than n* now does not follow from density anymore but would rather be predicted by it. The absence of such an interpretation for *at least n*, on the other hand, does now follow from density. This is unexpected and problematic because the account for *more than n* in positive environments is lost in negative ones. Moreover, it seems that a generalization is missed: as I already argued in Section 3 above, in the same environments where numerals with comparative modifiers do not have an exact-interpretation, numerals modified with superlative modifiers do not do so either. And the same holds for the environments where an exact-interpretation is available. In a way this is to be expected given that comparative *not more than n* can be replaced by superlative *at most n* and superlative *not at least n* can be replaced by *fewer than n* (but cf. Geurts and Nouwen 2007). This immediately predicts that we should observe a parallel behavior for *fewer than n* and *at most n* when embedded under negation. The former is equivalent to *at least n*, whereas the latter can be expressed by *more than n*. Neither of them appears to have the exact-interpretation: (49) does not imply that Jack read exactly four books. Again, (49) is not perfect given the positive polarity status of superlative *at most*. And (50) does not imply that Jack read exactly three books. Crucially, Fox and Hackl (2006) would predict the latter.

(49) ?Jack didn't read at most three books.
$\not\rightsquigarrow$Jack read at most four books.

(50) Jack didn't read fewer than three books.
$\not\rightsquigarrow$Jack read fewer than four books.

Finally, to complete the empirical picture, observe that scalar implicatures reappear under necessity modals for both *at most n* and *fewer than n*, whereas such inferences are absent under existential modals:

(51) a. Jack is required to read at most three books.
$\rightsquigarrow$ Jack is not required to at most two books.
b. Jack is allowed to read at most three books.
$\not\rightsquigarrow$Jack is not allowed to read at most two books.

(52) a. Jack is required to read fewer than three books.
$\rightsquigarrow$ Jack is not required to read fewer than two books.
b. Jack is allowed to read fewer than three books.
$\not\rightsquigarrow$Jack is not allowed to read fewer than two books.

Embedding under negation has been shown to be problematic for Fox and Hackl (2006), as density does not make the correct predictions for *more than n* in such environments. It predicts no implicature. It must, however, be noted that Nouwen (2008: Section. 6.2) adduces evidence in favor of a density-based approach by looking at the complex numeral *no more than n*. He notes that there is a difference between the examples in (53): *no more than n* lends itself much more easily to an exact-interpretation than *not more than n* does. The former thus behaves as predicted by Fox and Hackl (2006).

(53) a. Jack didn't read more than three books.
$\not\rightsquigarrow$ Jack read more than two books.
b. Jack read no more than three books.
$\rightsquigarrow$ Jack read exactly three books.

Nouwen derives the difference in (53) by assuming that sentential negation *not* always has widest scope – in particular, wider scope than the exhaustivity operator leading to strengthened readings (cf. Subsection 6.1.3 below) – whereas *no* in *no more than n* does not. If this much is guaranteed, the strengthened reading of (53a) will be derived without taking negation into account. The result is, of course, contradictory given density. In the case of (53b), however, negation is part of the material strengthened, and thus density predicts a scalar implicature. The problem for such an approach is to account for the most salient reading of (54), where *only* has widest scope saying that Mary is the only person who did not read *Anna Karenina*. This reading should not be possible given Nouwen's assumptions, as sentential negation has obligatorily widest scope.

(54) Only Mary didn't read *Anna Karenina*.

Thus, while the facts seen with *no more than n* are interesting in their own right, they do not lend strong support to Fox and Hackl's (2006) approach given that the stipulation regarding obligatory widest scope for sentential negation does not seem to be warranted given (54) and many similar data. Therefore the problem discussed in this section does not disappear (as acknowledged to some extent by Nouwen 2008 himself). I must also add that it is not clear what should be responsible for the different status of (53a) and (53b). I can only speculate that it must be somehow connected to the fact that *no more than n* is a lexically complex item, whereas *more than n* obviously does not form a lexical item together with sentential negation. Here it should also be noted that we should expect the more compositional nature of (53a) to be a more reliable window into what is really going on with modified numerals under negation than the lexically somewhat idiosyncratic *no more than n*.

I thus conclude that Fox and Hackl's (2006) approach is problematic nonetheless.

6. The limits of two related accounts

Consider the sentences in (55a) and (55b). When are such sentences typically uttered by a speaker?

(55) a. At least three boys left.
b. More than three boys left.

One particularly salient type of context where such sentences can be uttered felicitously is the one where the speaker does not know how many boys exactly left. Assume otherwise. In particular, assume that the speaker knows that exactly four boys left. In such a situation the basic interpretation of both (55a) and (55b) would be felicitous. Nevertheless one would probably judge the speaker to be not very cooperative because the utterances in (55) would be slightly misleading. In other words, upon hearing one utter (55a), the hearer draws the ignorance inference that the speaker fails to believe that exactly three boys left. In fact, it seems that (55a) has the ignorance inference that for any number *n* larger than three the speaker fails to believe that *n*-many boys left. There are two ways to account for this I can think of. Unfortunately, both run into problems of their own. Let me discuss each of the approaches in turn now.

For concreteness let us make the rather uncontroversial assumption for the following discussion that the interpretations of sentences (55a) and (55b) involve existential quantification as in (56a) and (56b), respectively.[16]

(56) a. $[[(55a)]]^w = \exists x[|x|\geq 3 \wedge \text{boy}_w(x) \wedge \text{left}_w(x)]$
b. $[[(55b)]]^w = \exists x[|x|>3 \wedge \text{boy}_w(x) \wedge \text{left}_w(x)]$

6.1 exactly n *alternatives*

Given the descriptive generalization argued for, one might suspect that the following is a plausible account of the data discussed. One might want to argue that it is literally *exactly-n*-alternatives that create a problem for the generation of scalar implicatures. In this subsection I show that such an account faces at least one big problem under a Gricean theory of scalar implicatures. For reasons of space, I will only show how such an account would work for numerals modified by superlative *at least*. The problem shows up for comparative modifiers as well, though.

Assume that *at least n* has two types of alternatives. It has alternatives of the form {exactly *n, exactly n+1, exactly n+2*, . . .}. Moreover it has scalar alternatives where only the numeral is replaced by another, stronger numeral. The alternatives for the sentence in (57) are then as in (58).

(57) At least three boys left.

(58) *Alt*([[At least three boys left]]) = {exactly 3 boys left, exactly 4 boys left, . . . , at least 3 boys left, at least 4 boys left, . . .}

Note that the alternatives in (58) are partially ordered by entailment. All the alternatives entail the basic meaning of (57) – that is, all entail that at least three boys left. Let us see how the neo-Gricean reasoning described in Subsection 2.1 for the strengthening of bare numerals would handle the case of (57).

A hearer of (57) draws the basic inference in (59a). That is, given the maxim of quality the hearer concludes that the speaker believes the plain meaning of (57), which says that at least three boys left. Employing the maxim of quantity, the hearer reasons about the strictly stronger alternatives, which would have been relevant for the discussion: if the speaker believed that exactly three boys left or that exactly four boys left (and similarly for any higher numeral), the speaker would have said so. Since she did not say so, she does not believe these alternative propositions to be true. This derives the ignorance inference in (59b). In a completely parallel fashion, the hearer reasons about the derived scalar alternatives in (58). If the speaker had evidence that at least four boys left, she would have said so. She did not do so. Therefore she does not believe that the stronger scalar alternatives are true, (59c).

(59) a. *Basic inference of* (57): B_S(that at least 3 boys left)
b. *Ignorance inference of* (57):
$\neg B_S$(that exactly 3 boys left) $\wedge$ ¬BS (that exactly 4 boys left) $\wedge$ . . .
c. *Scalar inference of* (57):
$\neg B_S$(that at least 4 boys left) $\wedge$ ¬BS (that at least 5 boys left) $\wedge$ . . .

The next question is whether any of the inferences in (59) can be strengthened. This is only possible if no contradiction arises. First, the hearer will not derive the stronger inference that the speaker believes it to be false that exactly three boys left. Otherwise it would entail together with the basic inference that the speaker believes it to be true that at least four boys left. But this contradicts the scalar inference in (59c), which says that the speaker does not believe that at least four boys left.

How about the scalar inference in (59c) – can it be strengthened? If the speaker believed that it is false that at least four boys left, she would have to believe that exactly three boys left, given the basic inference. But this contradicts the ignorance inference in (59b). It can be seen that we are essentially facing the symmetry problem discussed in Subsection 2.1: the hearer can neither conclude that the speaker believes it to be false that exactly three boys left nor that she believes it to be false that at least four boys left. But this also means that we have (almost) solved our initial problem. Sentences with non-embedded *at least n* do not appear to give rise to scalar implicatures. The strengthened interpretation derived so far is then as in (60).

(60) *Strengthened interpretation of* (57):
B_S(that at least 3 boys left) $\wedge$
$\neg B_S$(that exactly 3 boys left) $\wedge$ $\neg B_S$(that at least 4 boys left)

6.1.1 Reappearance of scalar implicatures Recall that scalar implicatures of modified numerals appear when embedded under universal quantifiers, but not when embedded under existential ones, as first noted by Fox and Hackl (2006). In terms of our descriptive generalization this means that a scalar implicature appears whenever the embedding operator has the effect that the strengthened interpretation is not entailed by the basic interpretation of a minimally differing sentence with an *exactly n* expression. Consider the examples with *at least n* from above:

(61) Jack is required to read at least three books.
$\rightsquigarrow$ Jack is not required to at least read four books

(62) Jack is allowed to read at least three books.
$\not\rightsquigarrow$Jack is not allowed to read at least four books

The relevant alternatives for (61) are as in (63).

(63) *Alt*([[Jack is required to read at least three books]]) = {Jack is required to read exactly 3 books, Jack is required to read exactly 4 books, . . ., Jack is required to read at least 3 books, Jack is required to read at least 4 books, . . .}

A strengthened interpretation of (61) where the ignorance and scalar inferences have been further strengthened as in (64) is possible. It is non-contradictory for the speaker to believe that Jack must read three books while at the same time believing that he neither must read exactly three nor at least four books.

(64) *Strengthened interpretation of* (61):
B_S(that Jack is required to read at least 3 books) $\wedge$
$B_S\neg$(that Jack is required to read exactly 3 books) $\wedge$
$B_S\neg$(that Jack is required to read at least 4 books)

The alternatives for (62) are as in (65).

(65) *Alt*([[Jack is allowed to read at least three books]]) =
{Jack is allowed to read exactly 3 books, Jack is allowed to read exactly 4 books, . . ., Jack is allowed to read at least 3 books, Jack is allowed to read at least 4 books, . . .}

This time a strengthened interpretation saying that the speaker believes that Jack is allowed to read at least three books but also believes that he is neither allowed to read exactly three books nor to read at least four books is contradictory – that is, the ignorance and scalar inferences cannot be further strengthened. The strongest interpretation possible is thus the one in (66).

(66) *Strengthened interpretation of* (62):
B_S(that Jack is allowed to read at least 3 books) ∧
¬B_S(that Jack is allowed to read exactly 3 books) ∧
¬B_S(that Jack is allowed to read at least 4 books)

A parallel account can be given for *more than n*. Therefore an approach making use of *exactly n* alternatives correctly accounts for the reappearance of scalar implicatures under certain operators. Spector (2005) offers an account to the present problem along the lines discussed just discussed. He proposes that *more than n* actually has the same formal alternatives as the disjunction *n + 1 or more*. As is well known the symmetry problem arises in disjunction independently. I will now turn to a problem. It is both a problem for Spector's more limited account and for the more general version discussed above.[17]

6.1.2 A problem What about the alternative propositions with numerals larger than 4? So far we have ignored the ignorance and scalar inferences for higher numerals. Notice that both the non-strengthened ignorance inference and the scalar inference for the sentence *At least three boys left* in (67), repeated from (59) above, contain further conjuncts that are said to not be believed by the speaker.

(67) a. *Basic inference of* (57): B_S(that at least 3 boys left)
b. *Ignorance inference of* (57):
¬B_S(that exactly 3 boys left) ∧ ¬B_S(that exactly 4 boys left) ∧ . . .
c. *Scalar inference of* (57):
¬B_S(that at least 4 boys left) ∧ ¬B_S(that at least 5 boys left) ∧ . . .

The question arises whether the hearer can conclude for any numeral *n* larger than 4 that the speaker has the belief that it is false that exactly *n*-many boys left.

The hearer cannot conclude that for all numerals *n* larger than 3 that the speaker believes that not exactly *n*-many boys left. Together with the basic inference that the speaker believes that at least three boys left, this would entail that she also believes it to be the case that exactly three boys left. But this contradicts the ignorance inference in (67b). But the hearer could conclude that for any numeral *m* larger than 4 the speaker believes it to be false that exactly *m*-many boys left. Note that this does not clash with the ignorance inference: the speaker is still allowed to not believe the propositions that exactly three boys left and that exactly four boys left to be true. Moreover, the hearer could then also conclude that the speaker believes the scalar alternatives with numerals larger than 4 to be false. In other words, it would follow that the speaker believes that fewer than five boys left. Together with the basic inference a strengthened interpretation would follow that says that the speaker believes that either exactly three or exactly four boys, but not more left, given in (68) with the problematic part in boldface. Clearly, this interpretation is not attested for (57).[18]

(68) *Strengthened interpretation of* (57):
B_S(that at least 3 boys left]) ∧
¬B_S(that exactly 3 boys left) ∧
¬B_S(that at least 4 boys left) ∧
B_S(that at least 5 boys left) ∧ B_S(that at least 6 boys left), ∧ . . .

A fully parallel problem would arise for a sentence with a numeral modified by comparative *more than*. This means that a scalar implicature is after all derived for modified numerals *at least n* and *more than n*, namely one stating that *not at least n* + *2* and one stating *not more than n* + *2* is the case, respectively. That is, it appears that the initial problem has just been shifted one level up on the number scale, and we have only partially accounted for the problem. It is important to see the nature of the problem a little clearer. The result that no scalar implicature for the numeral *n* + *1* is generated was obtained because the basic inference of the sentence (57) sets a lower bound on how many boys left, namely three. Because of this and the respective ignorance inference the scalar inference where *three* is replaced by the next larger numeral cannot be strengthened. But for scalar alternatives where *three* is replaced with a numeral at least as large as five this strategy does not work: the basic inference does not set a sufficiently high lower bound on how many boys left. Thus we see that a neo-Gricean framework following Sauerland (2004) makes wrong predictions for modified numerals when employing *exactly n* alternatives. I will now briefly show that a grammatical account of scalar implicatures faces a similar problem when using *exactly n* alternatives.

6.1.3 A grammaticality-based version The grammatical view of scalar implicatures (cf. Chierchia 2006, Chierchia *et al*. 2012; Fox 2007a) derives the strengthened interpretation of a sentence with a Horn-alternative in it by employing an exhaustivity operator O. That is, they assume that the strengthened interpretation of a sentence like *Jack read three books*, i.e., its exact-interpretation, is derived by applying an operator O such as (69) similar to *only* to the proposition denoted by the sentence, the prejacent (cf. Krifka 1995). This operator states that the prejacent is true and that all alternatives to the prejacent in the set C are either entailed by it or false. In other words, all alternatives to the proposition that Jack read three books not entailed by it must be false. As a consequence it must be false that Jack read four books or any larger number of books.

(69) $[[O]](C_{<<s,t>,t>})(p_{<s,t>})(w_s) = p(w) = 1 \wedge \forall q \in C[q(w) = 1 \rightarrow p \subseteq q]$

Following Fox (2007a), who in turn follows Groenendijk and Stokhof (1984) and Sauerland (2004), let us moreover adopt a version of the exhaustivity operator O that only negates those alternatives in C whose negation does not automatically require the truth of some other alternative in C. These alternatives are called the *innocently excludable* ones. What this does is to spare those alternatives from negation that would otherwise contradict each other. That is, the negation of these alternatives is not factored into the strengthened interpretation. The definition of O is as in (70) (cf. Fox 2007a: (61)).[19]

(70) $[[O]](C_{<<s,t>,t>})(p_{<s,t>})(w_s) = p(w) = 1 \wedge$
$\forall q \in C$ [q is innocently excludable given $C \wedge p \not\subseteq q \rightarrow q(w) = 0$]
(where q is innocently excludable given C if $\neg\exists q' \in C[p(w) \wedge \neg q(w) \rightarrow q'(w)]$)

If *At least 3 boys left* has the alternatives in (71), repeated from (58), a parallel problem to the one raised for the neo-Gricean account follows: negating, on the one hand, *exactly 3 boys left* would automatically include *at least 4 boys left* given the basic meaning of the sentence under discussion. Due to the basic meaning of the sentence, it would similarly follow that negating *at least 4 boys left* would automatically include *exactly 3 boys left*, because it would require that fewer than four boys left. Therefore the scalar implicature that exactly three boys left is not generated. Negating *at least 5 boys left*, on the other hand, does neither automatically include *at least 4 boys left* nor *exactly 3 boys left* (nor *exactly 4 boys left):* if fewer than five boys left, this is compatible with exactly three or exactly four boys leaving. Similarly, the negation of *exactly 4 boys left* would neither automatically include *at least 3 boys left* nor *at least 5 boys left* (nor *at least 4 boys left):* the negation of the non-monotonic *exactly 4 boys left* is compatible with at least three boys leaving – that is, exactly three boys leaving given the basic meaning of the sentence – and with at least five boys leaving. Therefore alternatives with a higher numeral than 4 can be negated deriving again the problematic reading in (72).

(71) *Alt*([[At least three boys left]]) = {exactly 3 boys left, exactly 4 boys left, . . ., at least 3 boys left, at least 4 boys left, . . .}

(72) that at least 3 boys left ∧ ¬ that at least 5 boys ∧ ¬ that at least least 6 boys left ∧ . . .

The problem discussed is thus rather theory-independent.

6.2 *Non-monotonic alternatives*

Consider now an account where *at least* and *more than* themselves come with lexical alternatives.[20] In particular, the following Horn-sets are proposed: {at *least, at most*} and {more *than, fewer than*}. It is important to see that the alternatives in these sets are not ordered by monotonicity.[21] This means that for the sentences in (73), the alternatives in (74a) and (74b) are derived, respectively.

(73) a. At least three boys left.
b. More than three boys left.

(74) a. *Alt*([[At least three boys left]]) = {at least 3 boys left, at least 4 boys left, . . ., at most 3 boys left, at most 4 boys left, . . .}
b. *Alt*([[More than three boys left]]) = {more than 4 boys left, more than 5 boys left, . . ., fewer than 4 boys left, fewer than 5 boys left, . . .}

The goal is to create a symmetry problem by employing the alternatives given so that no scalar implicature is derived. For concreteness let us assume the grammatical view of scalar implicatures.[22] The LFs assumed for (73a) and (73b) are as in (75a) and (75b), respectively.

(75) a. O [C [at least three boys left]]
b. O [C [more than three boys left]]

Consider what the assumptions just laid out do for the example in (73a). O asserts that the prejacent is true – that is, it is true that at least three boys left. Which of the alternatives in (74a) are innocently excludable? Consider first the alternative *at most 3 boys left*. Negation of that alternative would require that at least four boys left. That is, the alternative *at least 4 boys left* would be automatically true, i.e., included in the strengthened meaning. Therefore *at most 3 boys* is not innocently excludable. Negation of the alternative *at most 4 boys left* would require the truth of the alternative *at least 5 boys left*. Again, *at most 4 boys left* is not innocently excludable. For completely parallel reasons *at-most*-alternatives with numerals larger than 4 are not innocently excludable either.

Consider next the *at-least*-alternatives. Of course *at least 3 boys left* cannot be negated. This would contradict the prejacent, which is required to be true. The negation of *at least 4 boys left* would entail together with the truth of the prejacent that exactly three boys left. But this would require the truth of the alternative *at most 3 boys left*. And therefore *at least 4 boys left* is not innocently excludable. Similarly, the negation of *at least five boys left* would require the truth of the alternative *at most 4 boys left*. Therefore it is not innocently excludable. And the same holds for *at-least*-alternatives with numerals larger than 5. None of them is innocently excludable.[23]

None of the alternatives in (74a) is innocently excludable. To see clearer what O defined as in (70) achieves, notice that the negations of these alternatives would contradict each other. For instance, the negation of *at least 4 boys left* – that is, the proposition that fewer than four boys left – contradicts the negation of the alternative *at most 3 boys left* – that is, the proposition that more than three boys left. The negation of both alternatives cannot be true at the same time. O therefore does not negate them. In other words, they are not innocently excludable. But from this it follows that the strengthened interpretation of (73a) is equivalent to its basic interpretation. It does not have a scalar implicature.

For parallel reasons no implicature is generated for (73b) with comparative *more than n*, where the alternatives are as in (74b). The alternative *fewer than 4 boys left* can be negated given the basic interpretation of (73b). The negation says that at least four boys left, which is just the basic interpretation of (73b). That is, the negation forces the inclusion of *more than 3 boys left*, which is innocent. But *fewer than 5 boys left* cannot be negated. It would automatically make the alternative *more than 4 boys left* true. It is not innocently excludable. And the same holds for *fewer-than*-alternatives with larger numerals. They are all not innocently excludable.

The alternative *more than 4 boys left* cannot be negated. Doing so would automatically include the alternative *fewer than 5 boys left*. Similarly the alternative *more than 5 boys left* cannot be negated either. It would force the inclusion of the alternative *fewer than 6 boys left*. Again, the same holds for *more-than*-alternatives with numerals larger than 6. None of them is innocently excludable. Thus no scalar implicature is generated for (73b), and its strengthened interpretation is just its basic interpretation.

It should be noted that the negated versions of *more than n* and *at least n*, as well as the modified numerals *fewer than n* and *at most n* discussed in Subsection 5.2 work in a fully parallel way to the cases discussed here. They will not exhibit any scalar implicatures either.

6.2.1 Reappearance of scalar implicatures Let us now look at embedded modified numerals with the examples in (76) and (77) repeated once more from above.

(76) Jack is required to read at least three books.
$\rightsquigarrow$ Jack is not required to read at least four books

(77) Jack is allowed to read at least three books.
$\not\rightsquigarrow$ Jack is not allowed to read at least four books

The symmetry problem does not arise for (76). The relevant alternatives are as in (78).

(78) *Alt*([[Jack is required to read at least three books]]) =
{Jack is required to read at least 3 books, Jack is required to read at least 4 books, . . ., Jack is required to read at most 3 books, Jack is required to read at most 4 books, . . .}

Can the *at-most*-alternatives be innocently excluded? The negation of the alternative *Jack is required to read at most 3 books* says that there is a world where Jack reads more than three books. In the example (73a) discussed above negation of the *at most 3* alternative automatically led to the inclusion of the *at least 4* alternative. Not so in this case: if there is a world where Jack reads more than three books, it does not follow that in all worlds Jack reads more than three books, as would be required by inclusion of the alternative *Jack is required to read at least 4 books*. Similarly, negation of *Jack is required to read at most 4 books* does not lead to the inclusion of the alternative *Jack read at least 5 books*. Parallel considerations apply for the remaining *at-most*-alternatives.

Consider next the *at-least*-alternatives. Negation of *Jack is required to read at least 3 books* is prohibited by the basic meaning of (76). *Jack is required to read at least 4 books*, however, can be innocently excluded. In particular its negation does not lead to the automatic inclusion of *Jack is required to read at most 3 books*. If there is a world where Jack reads fewer than 4 books – as the former requires –, it does not follow that in all worlds Jack reads fewer than 4 books – as the latter requires. The same holds for alternatives with larger numerals. We thus get the strengthened interpretation for (76) in (79), which is the desired outcome.

(79) [[Jack is required to read at least three books]]^{S} =
Jack is required to read at least 3 books $\wedge$ for any $n > 3$ he is not required to read at least n-many books $\wedge$ for any $m \geq 3$ he is not required to read at most m-many books

Consider now (77), which has the alternatives in (80).

(80) *Alt*([[Jack is allowed to read at least three books]]) = {Jack is allowed to read at least 3 books, Jack is allowed to read at least 4 books, . . ., Jack is allowed to read at most 3 books, Jack is allowed to read at most 4 books, . . .}

Negating the alternative *Jack is allowed to read at most 3 books* automatically includes the alternative *Jack is allowed to read at least 4 books*. The negation of the former states that there is no world where Jack reads at most three books, i.e., in all worlds he reads at least four books. Thus the latter must be true. Similar considerations hold for the other *at-most*-alternatives.

Negating the alternative *Jack is allowed to read at least 4 books* would mean that in all worlds Jack reads fewer than four books. Therefore the alternative *Jack is allowed to read at most 3 books* would have to be true. In sum, none of the alternatives is innocently excludable, and the strengthened interpretation of (77) is equivalent to its basic meaning. In other words, the present theory correctly distinguishes between (76) and (77).

It is fairly easy to see that a parallel account for numerals modified by comparative *more than* embedded under universal or existential quantifiers can be given. Moreover, the remaining operators discussed in Subsection 3.2 that do lead to scalar implicatures are treated on a par by the present account, which might not always be desirable, unless something is said in addition. Let me turn to this problem.

6.2.2 Another problem Consider again the example with distributive conjunction of individuals, (81) repeated from (20a).[24]

(81) John and Mary both wrote more than three books.
$\rightsquigarrow$ Not both of John and Mary wrote more than four books

It has the relevant alternatives in (82).

(82) *Alt*([[John and Mary both wrote more than three books]]) =
{John and Mary both wrote more than 4 books, John and Mary both wrote more than 5 books, . . . , John and Mary both wrote fewer than 4 books, John and Mary both wrote fewer than 5 books, . . .}

The alternative *John and Mary wrote fewer than 4 books* is innocently excludable. Its negation says that either John or Mary wrote at least four books. This does not contradict the basic meaning of (81). But it also does not require any other alternative to be true, which is to be expected given what was shown in the preceding subsection. The same holds for any alternative of the form *John and Mary wrote fewer than n books* with $n \geq 4$. But then the strengthened interpretation of (81) amounts to (83), which requires that one of John and Mary wrote at most five books and for any numeral *n* at least as high as 4 one wrote at least *n* books. In other words, one of John and Mary must have read an infinite amount of books.

(83) [[John and Mary both wrote more than three books]]^{S} =
John and Mary both wrote more than 3 books, for any $n \geq 5$ either John or Mary wrote at most *n*-many books, for any $m \geq 4$ either John or Mary read at least *m*-many books

This is a serious problem, and it generalizes to modified numerals embedded under other universal operators. The problem might not appear to be as severe for (84) repeated from (76). But even here the strengthened interpretation given in (79) above will have the consequence that for any *n* larger than three Jack is allowed to read at least *n* books, an interpretation we would normally not associate with (84).

(84) Jack is required to read at least three books.
$\rightsquigarrow$ Jack is not required to read four books.

I do not see any particularly insightful way how to deal with this problem at the moment. It might be possible to restrict the set of alternatives. That is, if we restrict the set of available alternatives for (84) to alternatives with numerals between three and, say, nine, it would only follow that for any *n* larger than three and not larger than nine that Jack is allowed to read *n* books. In other words, Jack is allowed to read between three and ten books. Nothing is said about alternatives with larger numerals. Clearly, the requirement previously derived for (81) that one of John and Mary read an infinite number of books goes away, at least to some extent. Restricting the set of alternatives contextually might even make sense intuitively. When (84) is uttered, it does not appear that the alternative *Jack is required to read at least 100 books* is particularly relevant. But it is unclear to me at the moment how we can restrict the alternatives precisely in the way we want it so that no problems arise. I must leave this for future research. If this part can be further spelled out, then we might have reason to adopt an account of modified numerals with non-monotonic alternatives over one with *exactly n* alternatives, it seems.

7. Conclusion

In the present chapter I did the following things: first, I provided novel evidence suggesting that the varying absence and presence of scalar implicatures with numerals modified by comparative *more than* and superlative *at least* has very similar roots, if not the same root. I therefore argued against certain views on this topic found in the literature. Second, I suggested a new empirical generalization covering the data discussed. Third, I showed that existing analyses dealing with implicatures of modified numerals do not account for the full empirical generalization, as they only deal with one type of modified numerals. In particular, I argued that both a density-based account[25] and a focus-operator-based one make certain unwelcome predictions. Finally, I discussed and compared two related alternative approaches that could be taken to analyze the novel data collected. Both have their problems regardless of whether a neo-Gricean or a grammatical view of implicatures is taken. While the ultimate account of the puzzle discussed thus remains to be determined, it should, however, be noted that one of the two alternative accounts – that is, an account relying on non-monotonic alternatives for modified numerals – might be successful if an insightful way is found how the contextually relevant alternatives are to be restricted. I sketched what the necessary ingredient should look like. I therefore hope that this positive outlook provides opportunity for future research on this issue.

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NOTES

1. When talking about the neo-Gricean theory, I will refer to the meaning obtained before the scalar implicatures are factored in as the basic or the non-strengthened inference. The meaning where the implicatures are factored in is called the strengthened inference. When talking about the grammatical theory of scalar implicatures I will mostly replace "inference" by "meaning."
2. This step does not follow from Gricean reasoning as such. In order for it to go through, it is necessary to assume that a speaker is opinionated with respect to the truth of the alternatives of a given sentence (cf. the discussion by Sauerland 2004, Fox 2007a, but also Gazdar 1979).
3. For discussion of the symmetry problem see Fox (2007a) (also cf. Kroch 1972). The symmetry problem arises whenever one inference prohibits the strengthening of another inference, and vice versa.
4. Breheny (2008) casts doubt on the significance of data similar to (6). In particular, he shows that the inference can go in both directions depending on the context – that is, both upper- and lower-bounded implicatures are available with bare numerals in DE environments. This, of course, speaks against an implicature-based analysis of the exact-interpretation of bare numerals and in favor of a lexical analysis. Also cf. Kennedy's arguments in this volume in favor of a semantically driven approach to the exact-interpretation of numerals. I do not want to take a stance in this debate as it is orthogonal to the main argument of this chapter. In other words, even if the exact-interpretation of bare numerals should be accounted for by a lexical analysis, the puzzle to be discussed shortly remains.
5. There is one more type of account exemplified by Spector (2005) to be discussed in Section 6.
6. There are in fact more cases of modified numerals (for instance, *no more than n, between* n_i *and* n_j), classified by Nouwen (2010) into two subtypes. For discussion of *no more than n* see Subsection 5.2.
7. The inverse scope reading does not seem to be particularly salient in examples with conditionals where the modified numeral is in the consequent and should thus not interfere with the judgments. This is possibly due to a prohibition against inverse scope readings that would be stronger than the respective surface scope reading (Mayr and Spector 2011).
8. The inverse scope interpretation is nonsensical for (15a) and (15b) in normal contexts, because usually not everyone writes the same books. Similar considerations apply to examples below.
9. Geurts and Nouwen (2007), however, suggest that numerals modified by superlatives are not positive polarity items but rather modal expressions. Modal expressions, they argue, are more restricted in distribution than non-modal ones.
10. (22a) is acceptable under the inverse scope interpretation saying that at least three books are such that Jack did not read them. This is as expected from positive polarity items (cf. Szabolcsi 2004 a.o.) but tangential to the present discussion.
11. Together with Chierchia (2006); Chierchia *et al*. (2012); Fox (2007a) a.o., Fox and Hackl (2006) assume the so-called grammatical theory of scalar implicatures. See Subsection 6.1.3 below for discussion of how the negation of stronger alternatives works technically.

12. For arguments that *at least n* and *more than n* are inherently different see Geurts and Nouwen (2007); Nouwen (2008, 2010). For reasons of space I cannot discuss their proposal in more detail.
13. In order to account for scalar implicatures of *more than n* embedded under *every*, Fox and Hackl (2006) have to resort to an infinite domain of individuals over which the universal quantifier ranges, so that no contradiction arises under the density-based approach for the example (15a) (*Everyone wrote more than three books*) when the scalar implicature is factored in. The infinite set of dense degrees is distributed over the infinite set of individuals. How would they deal with scalar implicatures generated by *more than n* under conjunction as in (20a) (*John and Mary both wrote more than three books)* above, which can be seen to involve universal quantification of some sort? It appears that one has to assume that at the level where strengthening happens the universal conjunction ranges over an infinite set of individuals as well. That is, the restricted set provided by *John and Mary* must somehow be ignored at this level. Otherwise the implicature associated with (20a) would be blocked by density. Specific assumptions about granularity can guarantee this.
14. An anonymous reviewer points out that Fox and Hackl (2006) make the predictions with respect to *at least n* discussed here only insofar as the meaning for *at least n* given in the text is assumed. This is correct. Another semantics for *at least n* can always be developed. The challenge, however, is to have it behave in parallel to *more than n* given the discussion in the previous sections. But this latter fact is not straightforwardly derivable if the two types of modified numerals have completely independent meanings.
15. Two things should be noted for (43b) and (44b). The basic meanings given for them in (46a) and (48a), respectively, entail that Jack read at least one book. However, this does not seem to be required by either of them, as shown by the non-contradictory sentences in (i). Barwise and Cooper's (1981) Generalized Quantifier theory does not make this prediction (cf. the discussion by Krifka 1999).

    (i) a. Jack read at most three books, in fact he read none.
    b. Jack read fewer than three books, in fact he read none.

    This fact is, however, not problematic for the argument given in the text. For (46) observe that even if *at most 3* includes the possibility of zero books, it would still follow that Jack read $3 - \varepsilon$ books, which is compatible with the implicature that he read more than $3 - \varepsilon$ books. That is, exact-interpretation is predicted. Similarly, for (48): even if it were possible that Jack read zero books, it is true that he read $3 - \varepsilon$ books and therefore that he did not read $3 - \varepsilon/2$ books. The latter is at odds with the implicature, and it would thus not be generated. Second, the interpretations given do not guarantee that Jack did not read more than three books because of cumulativity, as Krifka (1999) shows.
16. What is more controversial is how these interpretations are compositionally derived. The generalized quantifier tradition, on the one hand, would argue that the modifier and the numeral form a constituent taking the NP as a restrictor (cf. Barwise and Cooper 1981, Keenan and Stavi 1986). Krifka (1999) and Geurts and Nouwen (2007), on the other hand, claim that the NP and the numeral form a constituent. The modifier only supplies the modification of the numeral. The existential import is derived from existential closure. This latter option needs some additional

assumptions. I do not want to take a side in this debate. Christopher Kennedy in this volume has some related discussion.

17. Also see Fox (2007b) for another problem in relation with Spector's account.
18. For readability, the strengthened ignorance inferences for numerals larger than 4 have been left out. That is, the inference that the speaker believes it to be false that exactly five boys left has been left out, and the same for larger numerals.
19. Actually Fox (2007a) shows that the definition of *innocent exclusion* should make reference to that subset of C that only contains propositions that are non-weaker than p. For present purposes it seems to me that (70) suffices.
20. I especially thank Uli Sauerland (p.c.) for helpful discussion on this subsection.
21. This contradicts Matsumoto's (1995) assumptions that the fundamental condition on Horn-sets is that the alternatives are ordered by monotonicity.
22. Nothing said below really hinges on this. The neo-Gricean view could also be adopted. Given that the alternatives employed are not ordered by strength some complications might arise that could, for instance, be overcome by employing the maxim of quality instead of the maxim of quantity (cf. Sauerland 2012).
23. Note that there are in fact also alternatives with numerals smaller than 3. Consider the alternative *at least 2 boys left*. It is entailed by the basic interpretation of (73a). By the semantics of O in (70) it therefore cannot be negated. The same holds of course for *at least 1 boy left*. Consider next the negation of the alternative *at most 2 boys left*. It forces the inclusion of the alternative *at least 3 boys left,* which is just the basic meaning of the sentence. Thus it is innocently excludable. The same holds for *at most 1 boy left*. Thus these alternatives do not affect the overall result obtained for (73a).
24. I thank an anonymous reviewer for pointing out this problem, which I had noticed independently after having submitted the paper. I was unable to find a fully satisfying solution to this issue.
25. This raises the question of whether the universal density of measurement is needed at all for natural language semantics. Fox and Hackl (2006) argue that it is, for instance, also needed to account for weak island phenomena. If Abrusán and Spector (2011), however, are on the right track, then there might be a density-independent way to analyze weak islands.